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Exponential and Continued Fractions*

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We show that the simple continued fractions for the analogues of

coming from function field analogues of hypergeometric functions

1. INTRODUCTION

The continued fraction expansion of a real number is a fundamental and revealing expansion through its connection with Euclidean algorithm and with "best" rational approximations (see [HW]). At the same time, it is very poorly understood for some interesting numbers. We know that it is essentially unique and finite (i.e., terminating) exactly for rational numbers and periodic exactly for quadratic irrationalities. But apart from that, the expansion of even a single additional algebraic number is not explicitly known; we do not know even whether the partial quotients are unbounded for such numbers. (See [BS] for the function field situation).

For transcendental numbers of interest, it is not clear when to expect a continued fraction with a good "pattern". For example, Euler gave a nice continued fraction for e

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, ..., 1, 2n, 1, ...].$$

On the other hand, nobody has made any sense out of the pattern for π. (We restrict our attention to simple continued fractions: of course, there are many generalized continued fractions with nice patterns for numbers

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related to π). There is a vague folklore statement that the nice patterns come from the connection with hypergeometric functions and the generalized continued fractions for hypergeometric functions (see [P] sections 59 and 64) due to Gauss. (For more on this aspect and a survey, see [CC]).

Various fruitful analogies between the number fields and function fields of one variable over finite fields suggest exploring the question of finding "interesting patterns for interesting numbers" in the function field setting. In [T1], we found an interesting continued fraction for an analogue of e for $\mathbf{F}_q[t]$ coming from the Carlitz–Drinfeld exponential.

Classically, building on Euler's continued fractions (we will use the abbreviation "CF" from now on for continued fraction) for $e^{2/n}$, Hurwitz proved [H, P] that linear fractional transformations of $e^{2/n}$, with integer coefficients, have CF's whose partial quotients eventually consist of a fixed number of arithmetical progressions. For example, after the first digit 2, the CF for e consists of 3 progressions $1+0n$, $0+2n$, $1+0n$. For $e^{1/5}-1/5$ one needs 62 arithmetic progressions!

In characteristic p, arithmetic progressions are periodic and hence will give rise to quadratic numbers, whereas the numbers we look at are transcendental. Nonetheless we will show below that analogue of the Hurwitz class of numbers have very different interesting patterns, indicating that though the patterns and proofs are quite different, for some reason the analogues do have nice patterns.

After recalling the background material in Sections 1–3, our main results are contained in Sections 4–6.

For a general exposition on function field arithmetic we refer to [GHR] and for exposition on classical continued fractions to [HW] or [P].

1. Background on Continued Fractions

The basic reference here is [HW] or [P].

1.1. We start by recalling some standard facts and notation. $[(a_i)] :=$ $[a_0, a_1, a_2, ...]$ denotes the continued fraction

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots}}$$

The a_i is called the ith partial quotient. The quantity a_n' denotes $[a_n, a_{n+1}, ...]$. By a tail of the CF $[(a_i)]$ we mean a_n' for some n.

1.1.1. Let

$$p_0 = a_0, \qquad q_0 = 1, \qquad p_1 = a_1 a_0 + 1, \qquad q_1 = a_1$$

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2}$$

then

1.1.2. $p_n/q_n = [a_0, ..., a_n]$,

1.1.3. $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$,

1.1.4. $[a_0, a_1, ...] = (a'_n p_{n-1} + p_{n-2})/(a'_n q_{n-1} + q_{n-2})$ and

1.1.5. $q_n/q_{n-1} = [a_n, ..., a_1]$.

1.1.6. For $x = [(x_i)]$ and $y = [(y_i)]$, $x'_n = y'_m$ for some m and n (i.e. tails agree) if and only if $y = (ax+b)/(cx+d)$ with $a, b, c, d \in \mathbf{Z}$ and $ad - bc = \pm 1$. We say that such x and y are equivalent.

1.2. For a real number a, the continued fraction for a is obtained by repeating the procedure of “taking the integral part to be the partial quotient and starting again with the reciprocal of the number minus its integral part (if nonzero)”. Except possibly for a_0, all a_i's are then positive integers. Any CF with $a_i \in \mathbf{Z}$ and $a_i > 0$ for $i > 0$ converges to a real number (CF for a converges to a) and the equality of two such CF's implies equality of the corresponding partial quotients, except for the ambiguity in the last digits in terminating case due to $n = (n-1) + 1/1$. Uniqueness is restored if we insist that the last digit is not 1 (except in the special case $a = 1$ when we take the first digit (which is also the last digit) to be 1), this condition is guaranteed anyway if we follow the procedure above to find the CF expansion.

1.3. Now we turn to the function field case. Let $\mathbf{F}_q$ be a finite field of cardinality q and of characteristic p. Let $A := \mathbf{F}_q[t]$, $K := \mathbf{F}_q(t)$, $K_\infty := \mathbf{F}_q((1/t))$ and let Ω be the completion of an algebraic closure of K_∞. Then A, K, K_∞, Ω are well-known analogues (see [GHR]) of $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$ respectively.

For $a \in K_\infty$, if we replace the notion of “integral part of a number” by the analogous polynomial part $\sum_{i=0}^{k} A_i t^i$ in the Laurent expansion $\sum_{i=-\infty}^{k} A_i t^i$ of a, the same procedure gives CF for a, with $a_i \in A$. This time there is no sign condition forced on a_i (such as they must be monic), in contrast to the positivity condition in 1.2. On the other hand, for $i > 0$, $a_i \notin \mathbf{F}_q$. Conversely, with these conditions there is convergence and uniqueness of CF.

Observe that $[a+1, b+1] = [a, 1, b]$ in characteristic 2.

2. Results of Euler and Hurwitz

The basic reference here is [P].

2.1. Generalizing the CF for e mentioned in the introduction, Euler showed (the overline in the notation indicates infinite arithmetic progressions), that for $n > 1$

$$e^{1/n} = \overline{[1, n-1+2in, 1]}_{i=0}^{\infty} = [1, n-1, 1, 1, 3n-1, 1, 1, 5n-1, 1, 1, ...]$$

and for odd $n > 1$,

$$e^{2/n} = \overline{[1, (n-1)/2+3in, 6n+12in, (5n-1)/2+3in, 1]}_{i=0}^{\infty}.$$

Hurwitz showed (see [P] for the full statement) that if you have a number whose CF consists of arithmetic progressions from some point onwards, then the same property holds for the number obtained by applying linear fractional transformation of any nonzero determinant to it. (Note that if the determinant is ± 1, then this follows from 1.1.6 already and if the determinant is zero, we get the degenerate case of rational numbers).

2.1.1. In particular, $(ae^{2/n}+b)/(ce^{2/n}+d)$ for n a positive integer and $a, b, c, d \in \mathbf{Z}$ with $ad-bc \neq 0$ have all CF's whose partial quotients are eventually in a fixed number of arithmetic progressions. But the process to write down the CF is quite involved and there is no easy "formula" in general.

2.1.2. We give two examples worked out by Hurwitz [H]:

$$2e = [5, 2, \overline{3, 2i, 3, 1, 2i, 1}]_{i=1}^{\infty}$$

$$\frac{e+1}{3} =$$

$$[1, 4, \overline{5, 4i-3, 1, 1, 36i-16, 1, 1, 4i-2, 1, 1, 36i-4, 1, 1, 4i-1, 1, 5, 4i, 1}]_{i=1}^{\infty}.$$

3. The Exponential for $\mathbf{F}_q[t]$

The basic reference here is [C1].

3.1.1. Let $[i] := t^{q^i} - t$. This is just the product of monic irreducible elements of A of degree dividing i. Note $[i+1] = [i]^q + [1] = [i] + [1]^{q^i}$.

3.1.2. Let $d_0 := 1$, $d_i := [i]\, d_{i-1}^q$, $i > 0$. (d_i is Carlitz' F_i.) This is the product of monic elements of A of degree i.

3.1.3. Let

$$e(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i}$$

This is the exponential for $\mathbf{F}_q[t]$, introduced by Carlitz [C1]. We put $e := e(1)$. For many analogies with the properties of the classical exponential, we refer the reader to the introduction to [T1].

4. The $\mathbf{F}_q[t]$ Case

4.1. We start by recalling the result of [T1] and pointing out the immediate consequences, even though they will also follow from our main results below:

Theorem 1. *Define a sequence x_n with $x_1 := [0, z^{-q}[1]]$ and if $x_n = [a_0, a_1, ..., a_{2^n-1}]$, then setting*

$$x_{n+1} := [a_0, ..., a_{2^n-1}, -z^{-q^n(q-2)} d_{n+1}/d_n^2, -a_{2^n-1}, ..., -a_1]$$

Then

$$x_n = \sum_{i=1}^{n} \frac{z^{q^i}}{d_i}$$

In particular, $e(z) = z + \lim_{x \to \infty} x_n$ and for $q = 2$,

$$e = e(1) = [1, [1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5], ...]$$

(*More explicitly, for $n > 0$ the n-th partial quotient is $t^{2^{u_n}} - t$ with u_n being the exponent of the highest power of* 2 *dividing* $2n$).

4.2. We now introduce some terminology to talk about such patterns, with negative reverse repetition: By a CF μ of pure e-type with the initial block $\vec{X} = (a_1, ..., a_{k_1})$ and digits w_i, we mean CF described by its suitable truncations μ_i as follows: Let $\mu_1 := [a_0, ..., a_{k_1}, w_1]$ and if $\mu_i = [a_0, a_1, ..., a_{k_i}, w_i]$ then

$$\mu_{i+1} := [a_0, a_1, ..., a_{k_i}, w_i, -a_{k_i}, -a_{k_i-1}, ..., -a_1, w_{i+1}]$$

Let's make it more visual: For $\vec{Y}=(y_1, ..., y_k)$, put $\overleftarrow{Y}=(y_k, ..., y_1)$ and $-\overleftarrow{Y}:=(-y_k, -y_{k-1}, ..., -y_1)$. Then we have $\mu_1=[a_0, \vec{X}, w_1]$ and for $\mu_i=[a_0, \vec{Y}, w_i]$ we have $\mu_{i+1}=[a_0, \vec{Y}, w_i, -\overleftarrow{Y}, w_{i+1}]$ so that

$$\mu=[a_0, \vec{X}, w_1, -\overleftarrow{X}, w_2, \vec{X}, -w_1, -\overleftarrow{X}, w_3, \vec{X}, ...]$$

We say that $x=[(x_i)]$ is of e-type, if it is equivalent (see 1.1.6) to some μ of pure e-type, i.e., for some n, the tail x'_n is a tail of some CF of pure e-type.

4.3. Theorem 1 then shows that e and $e(\alpha/f)-\alpha/f$ for $f\in A-\{0\}$ and $\alpha\in \mathbf{F}_q^*$ (see also 5.1, 5.2) are of pure e-type.

4.4. The proof of Theorem 1 (see also the remark following the proof in [T1]) is based on a calculation abstracted in the following lemma of [PS], [DMP] whose variants already appear in [S1].

Lemma 1. *Let* $\vec{X}=(a_1, ..., a_n)$, *so that* $[a_0, \vec{X}]=p_n/q_n$. *Then* $[a_0, \vec{X}, y, -\overleftarrow{X}]=p_n/q_n+(-1)^n/yq_n^2$.

Proof. This is a straightforward calculation using 1.1.1–1.1.5. ∎

The Main Result when $q>2$

Theorem 2. *Let* $q>2$, $a, b, c, d, f\in A$, $f\neq 0$. *If* $ad-bc\neq 0$, *then* $(ae(1/f)+b)/(ce(1/f)+d)$ *is of e-type. If* $a, b, d\neq 0$, *then* $M:=(a/b)\, e(1/f)+(c/d)$ *is of pure e-type.*

Proof. We first prove the second claim. We have

$$M=\left\{\left(\frac{c}{d}+\frac{1}{bf/a}+\cdots+\frac{1}{bf^{q^{n-1}}d_{n-1}/a}\right)+\frac{1}{bf^{q^n}d_n/a}\right\}+\frac{1}{bf^{q^{n+1}}d_{n+1}/a}+\cdots$$

Let $n-1$ be a positive integer greater than the degrees of a, b and d. Let the quantity (call it η) in the curly brackets $\{\ \}$ in the displayed equation have CF $[a_0, \vec{X}]$.

Claim. *The CF of M is of pure e-type with the initial block $\vec{X}$ and digits $w_i:=(-1)^{k_i}\, a(f^{q^{n+i-1}}d_{n+i-1})^{q-2}\,[n+i]/b$, for $i=1$ to ∞ (note that these are nonconstant integers, by* 3.1.2 *since* $q>2$), *where $k_i=1$ for $i>1$ and k_1 is the length of the initial block.*

Proof of the Claim. By 3.1.2, our choice of n implies that the quantity in the round backets ( ) in the displayed equation can be written with common (integral) denominator $bf^{q^{n-1}}d_{n-1}/a$, but it may not be the reduced denominator. On the other hand, since by our choice of n every prime dividing $bf^{q^n}d_n/a$ divides d_n or f, it will not divide the numerator of η written with the denominator $bf^{q^n}d_n/a$ showing that it is in fact the reduced denominator of η. Since $bf^{q^m}d_m/a = (bf^{q^{m-1}}d_{m-1}/a)^2\, a(f^{q^{m-1}}d_{m-1})^{q-2}\,[m]/b$, Lemma 1 finishes the proof of the claim and hence of the second claim in the Theorem by induction on m.

To deduce the first claim in the Theorem from this, it is enough, by analogue of 1.1.6 (see p. 601 of [BS]) to establish the following claim.

CLAIM. *Let $a, b, c, d \in A$, with $D := ad - bc \neq 0$ be given. Then there are $a', b', c', d' \in A$, with $a'd' - b'c' = 1$ and $r_1, r_2 \in K$ satisfying*

$$\frac{ax+b}{cx+d} = \frac{a'(r_1x+r_2)+b'}{c'(r_1x+r_2)+d'} \in K(x)$$

Proof of the Claim. We leave the easier case when one of a, b, c, d is zero to the reader. Let g be the greatest common divisor of a and c. Then $a' := a/g \in A$ and $c' := c/g \in A$ are relatively prime and hence there are b' and d' in A such that $a'd' - b'c' = 1$. With $r_2 := -(bg + b'D)/(a'D) \in K$, we have $b/d = (a'r_2 + b')/(c'r_2 + d')$. Hence we can solve for r_1, $t \in K$ such that $g = r_1 t$, $b = (a'r_2 + b')t$ and $d = (c'r_2 + d')t$. But this is equivalent to the displayed equation in the claim. This finishes the proof of the claim and hence of the Theorem. ∎

4.5. *Remark.* Notice that we have given, together with 1.1.6, an effective (finite) procedure to determine the pattern for analogues of Hurwitz numbers. Also, note that sometimes we can use a smaller size "building block" $\vec{X}$ by taking a smaller n than prescribed above (and keeping the rest of the recipe in the claim the same).

The case $q = 2$

4.6. When $q = 2$, the situation is more subtle. Now $-\overleftarrow{X} = \overleftarrow{X}$ and for $q = 2$ the expansion of e given in the Theorem 1 can be equally interpreted as negative reverse repetition or reverse repetition or just repetition. But these interpretations will lead to distinct generalization, as we will see shortly.

THEOREM 3. *Let $q = 2$, and let $a, b, c, d, f \in A$, $b, d, f \neq 0$. Let $M := (a/b)\, e(1/f) + c/d$. If b divides $t^2 + t$, then M is of pure e-type. If, in addition t divides b, (and does not divide a) then $N := (a/b)\, e(t/f) + c/d$ is also of pure e-type. If b is a square-free polynomial, then for infinitely many n, the CF for M is of the form $[a_0, \vec{X}, a_n, -\overleftarrow{X}, \ldots]$ with $\vec{X} = (a_1, \ldots, a_{n-1})$.*

Proof. We proceed exactly as in the case $q>2$. The fact that b divides d_n^{q-2} in that case may not carry over, but b still divides $[n+1]$, by 3.1.1 and we get the proof of the first claim. For the second claim, note that for large n, under the given condition both the terms in ( ) in the equation

$$bf^{2^{n+1}}d_{n+1}/at^{2^{n+1}}=(bf^{2^n}d_n/at^{2^n})^2\,([n+1]\,a/b)$$

are integral. Hence the claim follows exactly as before.

Now note that for large enough n, the truncated series (displayed in the proof of the last theorem) for M give convergents (i.e. truncated continued fractions) for M because of rapid convergence. (More precisely, if the truncation is p/q with $(p,q)=1$, then $\deg(M-p/q)<-2\deg q$, by straightforward calculation, and (e.g. by Lemma 1 of [BS]) this guarantees that p/q is a convergent). If b is square-free, let m be the least common multiple of the degrees of the primes dividing b. Then by 3.1.1, when m divides $n+1$, then b divides $[n+1]$ and Lemma 1 gives the reversal as claimed at the corresponding truncations. ∎

4.7. *Remark.* We can of course replace t by $t+1$ or t^2+t in the theorem above. If we look for the analogue of $2/n$ in the Hurwitz theorem, at first we may think of α/f with $\alpha\in\mathbf{F}_q$ (in particular, say $q-1$, which plays the role of 2, in many contexts). Indeed, since $e(\alpha/f)=\alpha e(1/f)$ such a result is included in our result. But it seems that the fact that 1 and 2 are the only allowed numerators in the classical case is related to the fact that only first and second roots of unity are in the ground field $\mathbf{Q}$. With this interpretation, if we look for the a-torsion points of the Carlitz module (see the introduction of [T1] or [GHR]; these are analogues of roots of unity), which are in K, then an easy calculation shows that for $q>2$ this forces $a\in\mathbf{F}_q$, but for $q=2$ $a=t$, $t+1$, or t^2+t are exactly the extra a's that are allowed. This fits with our result.

Now we show that the patterns in the general case are more subtle when $q=2$, giving a mixture of reversing and repeating of patterns, and we do not have the same results as for $q>2$. Since we have some general results in the previous theorem about square-free denominators and complete result for a degree one denominator, we now look at denominators with higher multiplicity and degrees.

THEOREM 4. *Let $q=2$. Then for $n>2$, and with $\overrightarrow{X_n}$ defined by $\sum_{i=0}^{n-2}1/(d_it^n)=[0,\overrightarrow{X_n}]$ we have,*

$$\frac{e}{t^n}=[0,\overrightarrow{X_n},t^{2^{n-1}-n},\overrightarrow{X_n},t^{2^n-n},\overrightarrow{X_n},t^{2^{n-1}-n},\overrightarrow{X_n},t^{2^{n+1}-n},\ldots]$$

Also, with $\vec{X}=(t^2+1,\, t,\, t+1)$ *we have*

$$\frac{e}{t^2}=[0,\, \vec{X},\, t^2,\, \vec{X},\, t^6,\, \vec{X},\, t^2,\, \vec{X},\, t^{14},\, \ldots].$$

Proof. We start with e/t^2: Observe that $[0, \vec{X}]=1/t^2+1/(t^2 d_1)$ and $[0, \vec{X}]+[0, \overleftarrow{X}]=1/t$.

LEMMA 2. *Let* $q=2$ *and* $\vec{X}=(a_1, \ldots, a_n)$, *so that* $[0, \vec{X}]=p_n/q_n$. *Then*

$$\begin{aligned}[0,\, \vec{X},\, y,\, \vec{X}] &= p_n/q_n+1/(q_n^2(y+(p_n+q_{n-1})/q_n))\\ &= [0,\, \vec{X}]+1/(q_n^2(y+[0,\, \vec{X}]+[0,\, \overleftarrow{X}])).\end{aligned}$$

Proof. The proof is a straightforward application of 1.1.1–1.1.5. In more detail

$$p_{n+1}/q_{n+1}:=[0,\, \vec{X},\, y,\, \vec{X}]=[0,\, \vec{X},\, y+p_n/q_n]$$

By 1.1.1 and 1.1.3, $p_{n+1}/q_{n+1}=p_n/q_n+(-1)^n/(q_n^2(a_{n+1}+q_{n-1}/q_n))$. But in our case, $a_{n+1}=y+p_n/q_n$. This proves the first equality. The second equality follows from 1.1.5. ∎

Let us write $p_{k_n}/q_{k_n}=1/t^2+\cdots+1/(t^2 d_n)=[0, \overrightarrow{Y_n}]$, it being understood as usual that p_{k_n} and q_{k_n} are relatively prime. Then $q_{k_n}=t^2 d_n$ is also the denominator of $q_{k_n-1}/q_{k_n}=[0, \overleftarrow{Y_n}]$ (by 1.1.5). Let our induction hypothesis be that $p_{k_n}/q_{k_n}+q_{k_n-1}/q_{k_n}=1/t$. This is true for $n=1$. Lemma 2 and 3.1.2 together imply that

$$[0,\, \overrightarrow{Y_n},\, t^{2^{n+1}-2},\, \overrightarrow{Y_n}]=p_{k_n}/q_{k_n}+1/(t^2 d_{n+1})=p_{k_{n+1}}/q_{k_{n+1}}$$

(notice that this is the claim in the statement of the Theorem, so we are really using double induction) and similarly also that

$$q_{k_{n+1}-1}/q_{k_{n+1}}=[0,\, \overleftarrow{Y_n},\, t^{2^{n+1}-2},\, \overleftarrow{Y_n}]=q_{k_n-1}/q_{k_n}+1/t^2(d_{n+1})$$

This shows that the induction hypothesis is true in general and consequently, the claim in the case of e/t^2 follows.

For the general case $n>2$,

CLAIM. $[0, \overrightarrow{X_n}]+[0, \overleftarrow{X_n}]=1/t^{n-1}$.

Assuming this, by induction using the lemma just as before we get the formula claimed for e/t^n, so it remains to prove the claim: We have

$q_{k_n} = t^n d_{n-2}$ and $p_{k_n} = d_{n-2}(1 + 1/d_1 + \cdots + 1/d_{n-2})$. Let $Q = td_{n-2} + p_{k_n}$ and $P = (1 + p_{k_n} Q)/q_{k_n}$. Then $Pq_{k_n} - Qp_{k_n} = 1$. We first show that P is an integer. Now

$$P = \frac{1 + p_{k_n}^2 + td_{n-2} p_{k_n}}{t^n d_{n-2}}$$

$$= \frac{\left(d_{n-2} + \dfrac{d_{n-2}}{d_1^2} + \cdots + \dfrac{d_{n-2}}{d_{n-3}^2}\right) + t\left(d_{n-2} + \dfrac{d_{n-2}}{d_1} + \cdots \dfrac{d_{n-2}}{d_{n-2}}\right)}{t^n}$$

By 3.1.1 and 3.1.2, (we calculate directly for $n = 3$ and check that P is an integer) it follows that for $n > 3$, $td_{n-2} \equiv 0 \pmod{t^n}$ and

$$\frac{d_{n-2}}{d_{n-i}^2} + t\,\frac{d_{n-2}}{d_{n-i+1}} = \frac{d_{n-2}}{d_{n-i+1}}([n-i+1] + t] = \frac{d_{n-2}}{d_{n-i+1}}\, t^{2^{n-i+1}} \equiv 0 \pmod{t^n}$$

Hence we see that P is an integer.

Now $Pq_{k_n} - Qp_{k_n} = 1$ whereas by 1.1.3, we have $p_{k_n - 1} q_{k_n} - q_{k_n - 1} p_{k_n} = 1$, so that $p_{k_n}(q_{k_n-1} - Q) = q_{k_n}(p_{k_n-1} - P)$. Now p_{k_n} and q_{k_n} are relatively prime by 1.1.3, and it follows easily from the definitions that the degree of $p_{k_n-1} - P$ is less than that of p_{k_n}. Hence $P = p_{k_n-1}$ and the latter formula proves the claim. Hence the proof of the claim and the theorem is complete. ∎

Next we deal with the CF for $e/\wp$ where $\wp = t^2 + t + 1$, the degree two prime. Let $\theta_i := \sum_{j=0}^{i} 1/(\wp d_j)$.

THEOREM 5. *The CF for $e/(t^2 + t + 1)$ is μ_∞, which is a limit of its truncations μ_i defined as follows:*

$$\mu_3 := [0, [1], [2]\,\wp, [1], [2][1] + 1, [2], [1], [2]]$$

For odd $k \geqslant 3$, if $\mu_k = [0, \vec{Y}]$, let $\mu_{k+1} := [0, \vec{Y}, [k+1]/\wp, \overleftarrow{Y}]$. For even $k \geqslant 4$, if $\mu_k = [0, \vec{Z}, [k]/\wp, \overleftarrow{Z}]$, let

$$\mu_{k+1} := [0, \vec{Z}, [k]/\wp, \overleftarrow{Z}, \lfloor [k+1]/\wp \rfloor, \overleftarrow{Z}, [k]/\wp, \vec{Z}]$$

where $\lfloor [k+1]/\wp \rfloor$ denotes the polynomial obtained as the quotient when $[k+1]$ is divided by $\wp$ using the division algorithm.

Proof. First observe that μ_k is well defined. It is enough to prove that $\mu_k = \theta_k$ for $k \geqslant 3$. The proof is by induction on k and holds for $k = 3$ by construction. For odd k the passage from k to $k+1$ follows by Lemma 1.

Now if we write $\mu_k = p_{n_k}/q_{n_k}$, then we simultaneously claim by induction that for odd k we have $(p_{n_k} + q_{n_k-1})/q_{n_k} = 1/\wp$. Again $k=3$ is a straightforward calculation. The induction on both claims is now complete by using the following lemma, in a similar manner as in the previous theorem. (The main calculation is spelled out after the proof of the Lemma).

LEMMA 3. *Let $q=2$ and $\vec{X}=(a_1, ..., a_n)$, so that $[0, \vec{X}] = p_n/q_n$. Then*

$$U := [0, \vec{X}, y, \overleftarrow{X}, x, \overleftarrow{X}, y, \vec{X}] = \frac{p_n}{q_n} + \frac{1}{yq_n^2} + \frac{1}{q_n^4 y^2 (x + (p_n + q_{n-1})/q_n)}$$

Proof. By Lemma 1 and 1.1.5, we have

$$\frac{p_m}{q_m} := [0, \vec{X}, y, \overleftarrow{X}] = \frac{p_n}{q_n} + \frac{1}{yq_n^2}, \qquad [0, \overleftarrow{X}, y, \vec{X}] = \frac{q_{n-1}}{q_n} + \frac{1}{yq_n^2}$$

Note that $q_m = yq_n^2$ and $p_m = 1 + yq_n p_n$. Also by 1.1.5, we have $q_{m-1} = p_m$ by the reversal symmetry of the CF. Hence we have, by a calculation as in Lemma 2,

$$\begin{aligned} U &= \left[0, \vec{X}, y, \overleftarrow{X}, x + \frac{q_{n-1}}{q_n} + \frac{1}{yq_n^2}\right] \\ &= \left(\frac{p_n}{q_n} + \frac{1}{yq_n^2}\right) + \frac{1}{q_m^2 \left(x + \dfrac{q_{n-1}}{q_n} + \dfrac{1}{yq_n^2} + \dfrac{q_{m-1}}{q_m}\right)} \end{aligned}$$

Substituting the values for q_m and q_{m-1} obtained above, we see that U is as claimed. This finishes the proof of the lemma.

Now we spell out the details of the application of the lemma: First let us write Q for the quantity $\lfloor [k+1]/\wp \rfloor$. By an easy induction we see that $t^{2^{2n+1}}$ is congruent to $t+1$ modulo $\wp$, so that $\wp Q = [k+1]+1$, for k even. Next note that $\theta_{k+1} = \theta_{k-1} + 1/(\wp d_k) + 1/(\wp d_{k+1})$. In the application of the lemma, p_n/q_n is θ_{k-1}, so that $q_n = \wp d_{k-1}$ and $1/yq_n^2 = 1/(\wp d_k)$ (by 3.1.2) and

$$\frac{1}{y^2 q_n^4 (x + (p_n + q_{n-1})/q_n)} = \frac{1}{\wp d_k^2 (\wp Q + 1)} = \frac{1}{\wp d_k^2 [k+1]} = \frac{1}{\wp d_{k+1}}$$

Hence the induction and the proof of the theorem is complete. ∎

These examples illustrate the subtleties of the case $q=2$. We have some more examples of this kind and in all these examples, we do find some inductive scheme of block reversal and block repetition. But we have not yet understood the situation fully for general Hurwitz type numbers when $q=2$. We hope to address that in a future paper.

5. Special Phenomena

There are some special phenomena in characteristic p, which do not seem to have analogues for real numbers:

5.1. For example $x=[(x_i)]$ implies $x^p=[(x_i^p)]$ in characteristic p, so nice patterns carry over for x^p from x.

5.2. For example, when $q=2$, $e(t^{1/2})-t^{1/2}$ has a pure e-type pattern as in section 4 as can be seen by putting $z=t^{1/2}$ in the Theorem 1. Similarly, if we subtract from $e(z)$ a few initial terms in the defining series we can get e-type patterns for resulting numbers for various z involving pth roots, for $q=2$ or even for general q. We leave it to the reader to find such variants.

5.3. Because of different kind of functional equations satisfied by exponentials (though they both correspond to the relevant cyclotomic theory, see [GHR]), care must be taken in comparison. For example, classically

$$\frac{ae^{2/n}+b}{ce^{2/n}+d}=\frac{ae^{1/n}+be^{-1/n}}{ce^{1/n}+de^{-1/n}}$$

whereas in our case

$$\frac{ae(1/f)+be(-1/f)}{ce(1/f)+de(-1/f)}=\frac{a-b}{c-d}$$

which may not be even defined if $c=d$. Similarly, we have good patterns for $e(a+1)-e(a)$, which is just e in disguise, but $e^{a+1}-e^{a}$ need not have good pattern. We leave it to the reader to find more examples combining these ingredients.

6. Complements

6.1. Lehmer [L] evaluates some CF whose partial quotients are in one or two arithmetic progressions in terms of Bessel functions. We proceed in roughly the same spirit to give some elegant CF's with elegant values: In [T2], a function field analogue of the hypergeometric series was introduced and a generalized CF was provided for it. We now give, in some special cases, the simple CF for values of these hypergeometric functions.

By specializing 3.6 of [T2], we get in notation of [T2], the following examples, when $q=2$:

(a) For integers $a>1$, ${}_1F_1(1; a; 1)^{q^{a-1}}$ is of pure e-type with the initial block d_{a-1} and digits $w_i := [a+i-1]$ for $i=1$ to ∞.

(b) For integers $a \geqslant 1$, $J_{a-1}(1)$ is of pure e-type, with the initial block d_{k-1} and digits $w_i := [k+i-1][i]^{q^{k-1}}$.

We leave it to the reader to provide more specializations of such type when $q \geqslant 2$ or when z is not necessarily one.

6.2. Objects which have patterns with "reversing and repeating" symmetries as in the examples of CF's above have been explored by various angles, such as with their connections with paperfoldings, functional equations for generating functions, Rudin–Shapiro sequence, ising model, automata, transcendence, geometrical representations, Toeplitz sequences etc. But we need an infinite alphabet in our case, in contrast to standard examples from automata theory. The reader is referred to the expository articles [AB], [DMP] and further references there.

6.3. *Remark.* It may interest the reader to know that the author first proved part of the function field results and then guessed (unaware of Hurwitz's results) that there should be patterns classically for analogous numbers and this was then verified on computers using Pari and Mathematica.

Note added in proof. The author has settled the patterns for $(ae(1/f)+b)/(ce(1/f)+d)$, when $q=2$. See the forthcoming paper by the author.

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